2024 Saturday Meetings' Report

In memory of Professor Hildebrando

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1 About the group

This group is open to anyone who wants to start learning or discussing math on Saturday mornings. Students include high school, undergraduate or graduate students. Every week, that week's volunteer student explains the selected topic and opens a discussion. The course is usually followed by a USP professor and the professor adds his own comments when necessary, but the goal is for volunteer students to work autonomously.

2 About the report

We decided to write this report for newcomers to follow the former discussions and to keep track of our progress. If there is something which is not clear, please contact us. There are some volunteers that can help in person.

3 To read

1.) Principles of Mathematical Analysis, Walter Rudin [1],
2.) Visual Complex Analysis, Tristan Needham [2].

4 2024 Group Report

4.1 Week 1

Discussion: There does not exist $p, q \in \mathbb{N}$ such that $\sqrt{2} = \frac{p}{q}$. (Rudin: Proof by contradiction).

Theorem 4.1. *Let $k, n \in \mathbb{N}$ such that $k, n \geq 2$. Then, either $\sqrt[k]{n} \in \mathbb{N}$ or $\sqrt[k]{n}$ is irrational. (Proved using the rational root theorem.)*

Theorem 4.2. *Let $r, s \in \mathbb{Q}, r \neq s$. Then, there is an irrational number between r, s.*

4.2 Week 2

Discussion: Ordered Sets.

Definition of supremum and infimum of a set. Least-upper-bound property.

4.3 Week 3

Discussion: Fields.

Definition 4.3. *A field is a set F with two operations, multiplications and addition, which satisfy field axioms.*

Discussion: Field axioms for (M):multiplication, (A): addition, (D): distribution law.

Example 4.4. *Is set $\mathbb{Z}$ of integers a field? Is set $\mathbb{Q}$ of rational numbers a field?*

Discussion: Ordered field.

4.4 Holiday topics that we covered

Complex numbers, metric spaces, topological space, a neighborhood of a point p, limit point of a set E.

4.5 Metric Spaces - Pages 30 and 31

When we work in the context of the real line, or even the real plane, it is natural to talk about the distance between numbers and vectors. For instance, given two numbers $a, b \in \mathbb{R}$, the distance from a to b is given by

$$|b - a| = |a - b|,$$

also known as the modulus of the two numbers, which can be defined in two equivalente ways[1]:

$$|x| = \sqrt{x^2}, \quad \text{or} |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases} .$$

When we study other sets, be them subsets of euclidean spaces or other kinds of sets, one natural question is "Can we talk about distance between objects that are not numbers?", and the short answer is yes! The long answer requires a definition, some theorems, and pretty much the remainder of chapter 2 from the book. To start the generalization, we begin with the main definition.

Definition 4.1. *Given a set X, a* ***metric on X*** *is a function $d : X \times X \to \mathbb{R}$ satisfying the three metric axioms for any two p, q in X (called* ***points*** *here):*

a) $d(p, q) > 0$ if $p \neq q$, and $d(p, p) = 0$;

[1]Proving that they are equivalent makes for a nice little exercise.

b) $d(p,q) = d(q,p)$;

c) $d(p,r) \leq d(p,q) + d(q,r)$, *(Triangular Inequality).*

The value of $d(p,q)$ *is called* ***the distance between p and q****.*

For instance, let's check these properties for two metrics
Example 4.5 (The Modulus of a Real Number)**.** *Given real numbers* x, y, z, *define* $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *by*

$$d(x,y) = |x - y|.$$

a) Since the modulus is, by definition, always positive, all that remains is to check if $d(x,x) = 0$. *In fact,*

$$d(x,x) = |x - x| = |0| = 0.$$

b) To show that $d(x,y) = d(y,x)$, *we must open up the two moduli and compare them. We have*

$$d(x,y) = |x-y| = \begin{cases} x-y, & x-y \geq 0 \\ y-x, & x-y < 0 \end{cases} = \begin{cases} x-y, & x \geq y \\ y-x, & x < y. \end{cases}.$$

On the other hand,

$$d(y,x) = \begin{cases} y-x, & y-x \geq 0 \\ x-y, & y-x < 0 \end{cases} = \begin{cases} y-x, & y \geq x \\ x-y, & y < x. \end{cases}$$

Looking at the two of them, it seems that they do in fact coincide, except when for the condition y=x - in which case both are 0 by the first item, hence all is well.

c) As for the triangular inequality, it is a matter of opening up the moduli of $|x-z|$, $|x-y|$, *and* $|y-z|$ *and checking/comparing individually for each case.*

Example 4.6. *For this next example, we'll define a metric that is available for absolutely any non-empty set, which is why it is known as the* ***discrete metric****. It works as counterexamples of a lot of things, and is an overall really nice example.*

Given a set $X \neq \emptyset$, *let* $d_0 : X \times X \to \mathbb{R}$ *be defined as*

$$d(p,q) = \begin{cases} 1, & p \neq q \\ 0, & p = q. \end{cases}$$

We'll show that this is in fact a metric, but as a brief comment before, we've started discussing metrics as a way of generalizing the notion of measurement of a distance - thus, a very natural question is "what does the discrete metric even measures?". The answer is that it is telling us a very basic answer, i.e. if two elements are equal or not.

Now to the proof, given p, q, r elements of X,

a) It follows that $d_0(p,q) = 1 > 0$ *if they are different, and, by definition,* $d_0(p,q) = 0$ *if p=q.*

b) When p and q are different, we have $d_0(p,q) = 1$ *and* $d_0(q,p) = 1$, *hence* $d_0(p,q) = d_0(q,p)$. *The case when they coincide follows readily, since everything will zero-out.*

c) If every point is the same (p=q=r), then $0 = 0 + 0$ *proves one case of the triangular inequality. If say p=q and* $q \neq r$, *then*

$$d_0(p,r) = 0 + d_0(q,r) = d_0(p,q) + d_0(q,r),$$

and the same holds by analogy when $q = r$ *and* $p \neq q$. *Finally, if* $p \neq q$, $q \neq r$, *and*

$p \neq r$, then all the distances will be one, hence

$$1 = d_0(p, r) \leq 2 = 1 + 1 = d_0(p, q) + d_0(q, r).$$

Therefore, d_0 is in fact a metric defined on the arbitrary non-empty set X.

Before delving deeper in the world of arbitrary metric spaces, it is important to give a few other definitions for Euclidean spaces ($\mathbb{R}^n$)

Definition 4.2. Given two real numbers a, b such that $a < b$, and a vector $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$, where x_i are real numbers for each $1 \leq i \leq k$, we say that

1) *the set $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is called **a segment of length b-a**;*
2) *an **interval of legth b-a** is the set $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$*
3) *a k-cell is the set of all points is $\mathbb{R}^k$ whose coordinates are within the values a_i and b_i, where $a_i < b_i$ for all $i = 1, 2, \ldots, k$, i.e., $\{x \in \mathbb{R}^k : a_i \leq x_i \leq b_i \forall 1 \leq i \leq k\}$*

With this definition, an interval is a 1-cell, a rectangle is a 2-cell, and so on.

Using the definition of a metric, one defines a few basic structures on a space. Most of them will be done on the next class, but we start with what is the general version of an interval and a segment outside the Euclidean set-up.

*Definition 4.3. Let X be a metric space endowed with the metric d. If $x \in X$ and r is a positive number, the **open ball B with center at x and radius r** is defined to be the set of all points whose distance to x is less than r. Mathematically, it is the set*

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

*We also define the **closed ball with center at x and radius r** by including the points whose distance to x is exactly r,*

$$\overline{B}_r(x) = \{y \in X : d(x, y) \leq r\},$$

*and the **circumference/boundary** of a ball (it works both for open and closed balls) as the set of points at a distance of exactly r from the center;*

$$\partial B_r(x) = \{y \in X : d(x, y) = r\}.$$

Balls in euclidean spaces, either closed or open, have the very nice property that you can take any two points inside the set, connect them by a line segment and that segment will be entirely contained within the ball. Such a property is known as *convexity*, formally defined by

*Definition 4.4. A subset $E \subseteq \mathbb{R}^k$ is called **convex** if*

$$\lambda x + (1 - \lambda) y \in E, \quad 0 < \lambda < 1$$

whenever x and y are in E.

Example 4.7 (Proof of Convexity of Balls)**.** *In this example, we prove the claim made above that balls are convex sets of euclidean spaces. To prove that, let $x, y, z \in \mathbb{R}^k$, $r > 0$, $0 < \lambda < 1$, and $y, z \in B_r(x)$. Since y and z are in the ball,*

$$|y - x| < r \quad \& \quad |z - x| < r,$$

from which we want to prove that $\lambda y + (1 - \lambda) z \in B_r(x)$ - in other words, our goal is to *prove that*

$$|\lambda y + (1-\lambda)z - x| < r.$$

As a matter of facts, we have

$$\begin{aligned}|\lambda y + (1-\lambda)z - x| = |\lambda y + (1-\lambda)z + \underbrace{\lambda x - \lambda x}_{=0} - x| &= |\lambda(y-x) + (1-\lambda)(z-x)| \\ &\leq \lambda|y-x| + (1-\lambda)|z-x| \\ &< \lambda r + (1-\lambda)r \\ &= \lambda r - \lambda r + r = r.\end{aligned}$$

Test Your Knowledge: Prove that closed balls and k-cells are convex sets.

4.6 Plan for the next lectures

03 August 2024 (For holiday group): Finite, Countable, Uncountable Sets

After holiday, we will study "Real Field" and finish Chapter 1.

References

[1] Walter Rudin et al. *Principles of mathematical analysis*, volume 3. McGraw-hill New York, 1964.

[2] Tristan Needham. *Visual complex analysis*. Oxford University Press, 2023.